Evolution and approximation in brittle fracture

Thermal dipping experiment Yuse-Sano 93

Bourdin08

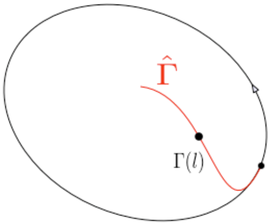
Evolution and approximation in brittle fracture

Multi-cracking Bourdin 06

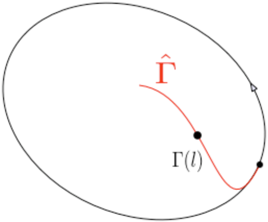
Evolution and approximation in brittle fracture

Multi-cracking Bourdin 06

Brittle Fracture à la **Griffith** 20

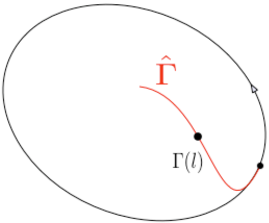


Brittle Fracture à la **Griffith** 20



preset crack path $\hat{\Gamma}$

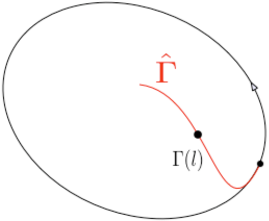
Brittle Fracture à la **Griffith** 20



preset crack path $\hat{\Gamma}$

crack of length l

Brittle Fracture à la **Griffith** 20



preset crack path $\hat{\Gamma}$

crack of length l

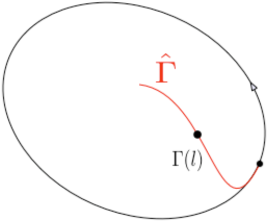
$E(t; u; l) :=$
$\int_{\Omega\setminus\Gamma(l)} W(\nabla\cdot)dx - \mathcal{F}(t,\cdot)$

elastic ↗ energy

work ↑ of loads

$u = g(t)$ on $\partial\Omega \setminus \Gamma(l)$

Brittle Fracture à la **Griffith** 20



preset crack path $\hat{\Gamma}$
crack of length l
$E(t;u;l) :=$
$\int_{\Omega\setminus\Gamma(l)} W(\nabla\cdot)dx - \mathcal{F}(t,\cdot)$

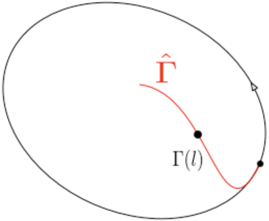
elastic ↗ energy   work   ↑ of loads

$u = g(t)$ on $\partial\Omega \setminus \Gamma(l)$

Quasistatic $\equiv$ elastic equilibrium at time t $\Rightarrow$

$$\mathcal{P}(t,l) := E(u(t,l),l) = \min_{u \ \text{k.a.}} E$$

Brittle Fracture à la **Griffith** 20



preset crack path $\hat{\Gamma}$

crack of length l

$E(t;u;l) :=$

$\int_{\Omega\setminus\Gamma(l)} W(\nabla\cdot)dx - \mathcal{F}(t,\cdot)$

elastic ↗ energy

work ↑ of loads

$u = g(t)$ on $\partial\Omega \setminus \Gamma(l)$

Quasistatic ≡ elastic equilibrium at time t ⇒

$$\mathcal{P}(t,l) := E(u(t,l),l) = \min_{u \ \text{k.a.}} E$$

Energy release rate: $G(t,l) := -\partial\mathcal{P}/\partial l(t,l)$

$$\text{Griffith} \Rightarrow \frac{dl}{dt}(t) \geq 0,\ G(t,l(t)) \leq k,\ (G(t,l(t)) - k)\frac{dl}{dt}(t) = 0$$

Problems

- crack path must be preset: how does a crack kink?
- initiation generically impossible:
- $\mathcal{P}$ concave in l $\quad\Rightarrow$ jump in crack growth: brutal growth

Reformulate Griffith

F-Marigo 98

$$\mathcal{E}(t;u;l) := \int_{\Omega\setminus\Gamma(l)} W(\nabla u)dx + kl - \mathcal{F}(t,u)$$

Reformulate Griffith

F-Marigo 98

$$\mathcal{E}(t;u;l) := \int_{\Omega\setminus\Gamma(l)} W(\nabla u)dx + kl - \mathcal{F}(t,u)$$

- Griffith's Model is equivalent to:
  - Unilateral stationarity: 1-parameter family of variations $l(t,\varepsilon) = l(t) + \varepsilon\hat{l}, \quad u(t,\varepsilon,l) = u(t,l) + \varepsilon v(t,l)$ $\Rightarrow \frac{d}{d\varepsilon}\mathcal{E}(t,u(t,\varepsilon,l(t,\varepsilon)),l(t,\varepsilon))\Big|_{\varepsilon=0} \geq 0$

    $\approx$ a necessary first order condition for minimality

Reformulate Griffith

F-Marigo 98

$$\mathcal{E}(t;u;l) := \int_{\Omega\setminus\Gamma(l)} W(\nabla u)dx + kl - \mathcal{F}(t,u)$$

- Griffith's Model is equivalent to:
  - Unilateral stationarity: 1-parameter family of variations $l(t,\varepsilon) = l(t) + \varepsilon\hat{l}, \quad u(t,\varepsilon,l) = u(t,l) + \varepsilon v(t,l)$ $\Rightarrow \frac{d}{d\varepsilon}\mathcal{E}(t,u(t,\varepsilon,l(t,\varepsilon)),l(t,\varepsilon))\Big|_{\varepsilon=0} \geq 0$

    $\approx$ a necessary first order condition for minimality
  - $l(t) \nearrow$ with t

Reformulate Griffith

F-Marigo 98

$$\mathcal{E}(t;u;l) := \int_{\Omega\setminus\Gamma(l)} W(\nabla u)dx + kl - \mathcal{F}(t,u)$$

- Griffith's Model is equivalent to:
  - Unilateral stationarity: 1-parameter family of variations
    $l(t,\varepsilon) = l(t) + \varepsilon\hat{l}, \quad u(t,\varepsilon,l) = u(t,l) + \varepsilon v(t,l)$
    $\Rightarrow \frac{d}{d\varepsilon}\mathcal{E}(t,u(t,\varepsilon,l(t,\varepsilon)),l(t,\varepsilon))\Big|_{\varepsilon=0} \geq 0$

    $\approx$ a necessary first order condition for minimality
  - $l(t) \nearrow$ with t
  - Energy balance:
    $$\frac{d}{dt}\mathcal{E}(t;u(t),l(t)) = \int_{\partial\Omega\setminus\Gamma(l(t))} DW(\nabla u(t))n\cdot\dot{g}(t)dS - \dot{\mathcal{F}}(t,u(t))$$
    $$= \int_{\Omega\setminus\Gamma(l(t))} DW(\nabla u(t))\cdot\nabla\dot{g}(t)dS - \dot{\mathcal{F}}(t,u(t))$$

A variational model à la **Mielke** 02

- Replace unilateral stationarity by global minimality

A variational model à la Mielke 02

- Replace unilateral stationarity by global minimality

$$\Vert$$

Expand test cracks

$$\Downarrow$$

- Global Stability:

$$\min_{u,\Gamma} \mathcal{E}(t,u,\Gamma) := \int_{\Omega\setminus\Gamma} W(\nabla u)\,dx + k\mathcal{H}^{N-1}(\Gamma) - \mathcal{F}(t,u)$$

$$\uparrow \qquad \nwarrow$$

$$\equiv g(t) \text{ on } \partial\Omega\setminus\Gamma \qquad \begin{cases} \Gamma\subset\overline{\Omega} \\ \Gamma\supset\cup_{s<t}\Gamma(s) \end{cases}$$

A variational model à la Mielke 02

- Replace unilateral stationarity by global minimality

$$\|$$

Expand test cracks

$$\Downarrow$$

- Global Stability:

$$\min_{u,\Gamma}\mathcal{E}(t,u,\Gamma) := \int_{\Omega\setminus\Gamma} W\left(\nabla u\right)dx + k\mathcal{H}^{N-1}(\Gamma) - \mathcal{F}(t,u)$$

$$\uparrow \qquad \nwarrow$$

$$\equiv g(t) \text{ on } \partial\Omega\setminus\Gamma \qquad \begin{cases} \Gamma\subset\overline{\Omega} \\ \Gamma\supset\cup_{s<t}\Gamma(s) \end{cases}$$

Looks like Mumford-Shah 89: for g datum,

$$\min_{u,\Gamma}\left\{1/2\int_{\Omega\setminus\Gamma}|\nabla u|^2dx + k\mathcal{H}^{N-1}(\Gamma) + \int_{\Omega}|u-g|^2dx\right\}$$

A variational model à la Mielke 02

- Replace unilateral stationarity by global minimality

$$\|$$

Expand test cracks

$$\Downarrow$$

- Global Stability:

$$\min_{u,\Gamma} \mathcal{E}(t, u, \Gamma) := \int_{\Omega \setminus \Gamma} W(\nabla u)\, dx + k\mathcal{H}^{N-1}(\Gamma) - \mathcal{F}(t, u)$$

$$\uparrow \qquad \nwarrow$$

$$\equiv g(t) \text{ on } \partial\Omega \setminus \Gamma \qquad \begin{cases} \Gamma \subset \overline{\Omega} \\ \Gamma \supset \cup_{s<t}\Gamma(s) \end{cases}$$

- Energy balance

A variational model à la **Mielke** 02

- Replace unilateral stationarity by global minimality

$$\Vert$$

Expand test cracks

$$\Downarrow$$

- Global Stability:

$$\min_{u,\Gamma} \mathcal{E}(t,u,\Gamma) := \int_{\Omega\setminus\Gamma} W(\nabla u)\,dx + k\mathcal{H}^{N-1}(\Gamma) - \mathcal{F}(t,u)$$

$$\uparrow \qquad \nwarrow$$

$$\equiv g(t) \text{ on } \partial\Omega\setminus\Gamma \qquad \begin{cases} \Gamma\subset\overline{\Omega} \\ \Gamma\supset\cup_{s<t}\Gamma(s) \end{cases}$$

- Energy balance

…. Immediate consequence: In a linear setting $(W(F)=\mu/2|F|^2)$ always initiation in finite time!

Time discretization

$I_n = \{0 = t_0^n, ..., T = t_{k(n)}^n\}, \nearrow I_\infty$ dense in $[0, T]$

- u_i^n, Γ_i^n minimizes $\int_{\Omega\setminus\Gamma} W(\nabla u)dx + k\mathcal{H}^{N-1}(\Gamma) - \mathcal{F}(t_i^n, u)$ with

$$\begin{cases} \Gamma_{i-1}^n \subset \Gamma \subset \overline{\Omega} \\ u = g_i^n \text{ on } \partial\Omega \setminus \Gamma \end{cases}$$

Time discretization

$I_n = \{0 = t_0^n, ..., T = t_{k(n)}^n\}, \nearrow I_\infty$ dense in $[0, T]$

- u_i^n, Γ_i^n minimizes $\int_{\Omega\setminus\Gamma} W(\nabla u)dx + k\mathcal{H}^{N-1}(\Gamma) - \mathcal{F}(t_i^n, u)$ with

$$\begin{cases} \Gamma_{i-1}^n \subset \Gamma \subset \overline{\Omega} \\ u = g_i^n \text{ on } \partial\Omega \setminus \Gamma \end{cases}$$

$$\Downarrow$$

- $\begin{cases} u^n(t) := u_i^n \\ \Gamma^n(t) := \Gamma_i^n \end{cases}$ on $[t_i^n, t_{i+1}^n)$

$$n \nearrow \infty \text{ ?}$$

Time discretization

$I_n = \{0 = t_0^n, ..., T = t_{k(n)}^n\}, \nearrow I_\infty$ dense in $[0, T]$

- u_i^n, Γ_i^n minimizes $\int_{\Omega\setminus\Gamma} W(\nabla u)dx + k\mathcal{H}^{N-1}(\Gamma) - \mathcal{F}(t_i^n, u)$ with

$$\begin{cases} \Gamma_{i-1}^n \subset \Gamma \subset \overline{\Omega} \\ u = g_i^n \text{ on } \partial\Omega \setminus \Gamma \end{cases}$$

- Mumford-Shah 89 + De Giorgi-Carriero-Leaci 89 $\Rightarrow$ Discrete weak formulation:

u_i^n minimizes $\int_\Omega W(\nabla u)dx + k\mathcal{H}^{N-1}(S(u) \setminus \cup_{j<i} S(u_j^n)) - \mathcal{F}(t_i^n, u)$ for all $u \in SBV(\mathbb{R}^N)$ with $u \equiv g_i^n$ outside $\overline{\Omega}$

The evolution

Thm (Dal Maso-Toader 02, F-Larsen 03, Dal Maso-F-Toader 05, Dal Maso ... 09):

- W C^1 with (or without) p-growth, p-coercive, convex or quasiconvex;
- Ω nice ;
- appropriate loads $\mathcal{F}(t,v)$ and displacements $g(t)$.

Then $\exists \Gamma(t) \nearrow, u(t) \in SBV,\ \nabla u \in L^p$ st

• $u(t)$ minimizes $\int_\Omega W(\nabla v)dx + k\mathcal{H}^{N-1}(S(v) \setminus \Gamma(t)) - \mathcal{F}(t,v)$ with $u(t) \equiv g(t)$ on $\mathbb{R}^N \setminus \overline{\Omega}$

• $S(u(t)) \subset \Gamma(t)$

• $\mathcal{E}(t) := \int_\Omega W(\nabla u(t))dx + k\mathcal{H}^{N-1}(\Gamma(t)) - \mathcal{F}(t,u(t))$ satisfies

$$\frac{d}{dt}\mathcal{E}(t) = \int_\Omega DW(\nabla u(t)) \cdot \nabla \dot{g}(t)\, dx + \text{ terms coming from } \mathcal{F} \ \Box$$

The evolution

Thm (Dal Maso-Toader 02, F-Larsen 03, Dal Maso-F-Toader 05, Dal Maso ... 09):

- W C^1 with (or without) p-growth, p-coercive, convex or quasiconvex;
- Ω nice ;
- appropriate loads $\mathcal{F}(t,v)$ and displacements $g(t)$.

Then $\exists \Gamma(t) \nearrow, u(t) \in SBV,\ \nabla u \in L^p$ st

• $u(t)$ minimizes $\int_\Omega W(\nabla v)dx + k\mathcal{H}^{N-1}(S(v) \setminus \Gamma(t)) - \mathcal{F}(t,v)$ with $u(t) \equiv g(t)$ on $\mathbb{R}^N \setminus \overline{\Omega}$

• $S(u(t)) \subset \Gamma(t)$

• $\mathcal{E}(t) := \int_\Omega W(\nabla u(t))dx + k\mathcal{H}^{N-1}(\Gamma(t)) - \mathcal{F}(t,u(t))$ satisfies

$$\frac{d}{dt}\mathcal{E}(t) = \int_\Omega DW(\nabla u(t)) \cdot \nabla \dot{g}(t)\, dx + \text{ terms coming from } \mathcal{F} \ \square$$

No a priori measurability property of u with respect to t.....

The evolution

Thm (Dal Maso-Toader 02, F-Larsen 03, Dal Maso-F-Toader 05, Dal Maso ... 09):

- W C^1 with (or without) p-growth, p-coercive, convex or quasiconvex;
- Ω nice ;
- appropriate loads $\mathcal{F}(t,v)$ and displacements $g(t)$.

Then $\exists \Gamma(t) \nearrow, u(t) \in SBV,\ \nabla u \in L^p$ st

• $u(t)$ minimizes $\int_\Omega W(\nabla v)dx + k\mathcal{H}^{N-1}(S(v) \setminus \Gamma(t)) - \mathcal{F}(t,v)$ with $u(t) \equiv g(t)$ on $\mathbb{R}^N \setminus \overline{\Omega}$

• $S(u(t)) \subset \Gamma(t)$

• $\mathcal{E}(t) := \int_\Omega W(\nabla u(t))dx + k\mathcal{H}^{N-1}(\Gamma(t)) - \mathcal{F}(t,u(t))$ satisfies

$$\frac{d}{dt}\mathcal{E}(t) = \int_\Omega DW(\nabla u(t)) \cdot \nabla \dot{g}(t)\, dx + \text{ terms coming from } \mathcal{F} \ \square$$

No a priori measurability property of u with respect to t.....

• Does not work in linearized elasticity !!!! no co-area formula: however results in 2d for connected cracks by Chambolle 03

The trouble with global minimality

- Global minimization does not agree with dead forces:

$$\inf_u \left\{ \int_\Omega W(\nabla u)dx + k\mathcal{H}^{N-1}(S(u)) - \int_\Omega f \cdot udx \right\} = -\infty$$

The trouble with global minimality

- Global minimization does not agree with dead forces:

$$\inf_u \left\{ \int_\Omega W(\nabla u)dx + k\mathcal{H}^{N-1}(S(u)) - \int_\Omega f \cdot u dx \right\} = -\infty$$

$$\Downarrow$$

Replace global by local .... Which??? small add-length, L^p

The trouble with global minimality

- Global minimization does not agree with dead forces:

$$\inf_u \left\{ \int_\Omega W(\nabla u)dx + k\mathcal{H}^{N-1}(S(u)) - \int_\Omega f \cdot u dx \right\} = -\infty$$

$$\Downarrow$$

Replace global by local .... Which??? small add-length, L^p

- Not enough: Chambolle-Giacomini-Ponsiglione 08: no initiation

2d, hard device,
"connected cracks"
W strictly convex, $\mathcal{C}^1$,
p-growth, ψ elastic sol.

$\Rightarrow$

x point of weak singularity
iff, for some $\alpha > 1$
$\limsup_{r \downarrow 0} \frac{1}{r^\alpha} \int_{B(x,r)} |\nabla \psi|^p dx \leq C$.

Thm: If all points in $\overline{\Omega}$ are points of weak singularity (with a uniform bound), then $\exists l^*$ s.t. if $\mathcal{H}^{N-1}(\Gamma) < l^*$, then

$$\int_\Omega W(\nabla \psi)dx < \int_\Omega W(\nabla u^\Gamma)dx + k\mathcal{H}^{N-1}(\Gamma) \quad \square$$

↑ solution with Γ as crack

The trouble with global minimality

- Global minimization does not agree with dead forces:

$$\inf_{u} \left\{ \int_\Omega W(\nabla u)dx + k\mathcal{H}^{N-1}(S(u)) - \int_\Omega f \cdot u dx \right\} = -\infty$$

$$\Downarrow$$

Replace global by local .... Which??? small add-length, L^p

- Not enough: Chambolle-Giacomini-Ponsiglione 08: no initiation

2d, hard device,
"connected cracks"
W strictly convex, $\mathcal{C}^1$,
p-growth, ψ elastic sol.

$\Rightarrow$

x point of weak singularity
iff, for some $\alpha > 1$
$\limsup_{r\downarrow 0} \frac{1}{r^\alpha} \int_{B(x,r)} |\nabla\psi|^p dx \leq C$.

Thm: If all points in $\overline{\Omega}$ are points of weak singularity (with a uniform bound), then $\exists l^*$ s.t. if $\mathcal{H}^{N-1}(\Gamma) < l^*$, then

$$\int_\Omega W(\nabla\psi)dx < \int_\Omega W(\nabla u^\Gamma)dx + k\mathcal{H}^{N-1}(\Gamma) \quad \square$$

↑ solution with Γ as crack

$$\Downarrow$$

- ψ is local minimizer of the energy in any topology finer than L^1

The trouble with global minimality

- Global minimization does not agree with dead forces:

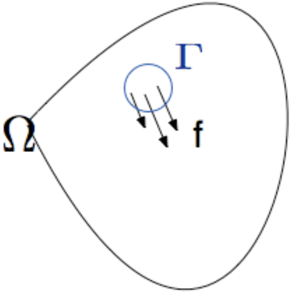
$$\inf_u \left\{ \int_\Omega W(\nabla u)dx + k\mathcal{H}^{N-1}(S(u)) - \int_\Omega f \cdot udx \right\} = -\infty$$

$$\Downarrow$$

Replace global by local .... Which??? small add-length, L^p

- Not enough:Chambolle-Giacomini-Ponsiglione 08: no initiation

- Does not cure dead forces: small crack Γ will kill potential energy.



Possible sol.: Non-interpenetration

The trouble with global minimality

- Global minimization does not agree with dead forces:

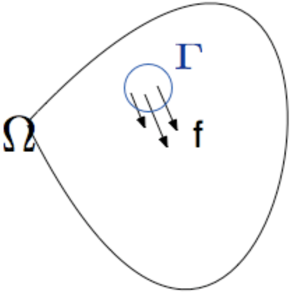
$$\inf_u \left\{ \int_\Omega W(\nabla u)dx + k\mathcal{H}^{N-1}(S(u)) - \int_\Omega f \cdot u dx \right\} = -\infty$$

$\Downarrow$

Replace global by local .... Which??? small add-length, L^p

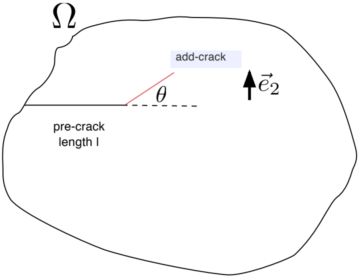
- Not enough: Chambolle-Giacomini-Ponsiglione 08: no initiation

- Does not cure dead forces: small crack Γ will kill potential energy.



Possible sol.: ~~Non-interpenetration~~

Kinking - the classics



- crack tip singularity:

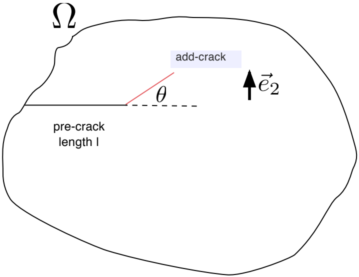
$$u = \sqrt{r}\sum_{i=1,2}\{K_i(t, l{+}l', \theta)\varphi_i\} + \hat{u}$$

with $\hat{u}$ smoother; φ_i universal fcts.

$=: u_{00}$(defined on all of $\mathbb{R}^2$) $+\ \hat{u}$

- $K_{1(2)} = 0$ if $\sigma\vec{e}_2 \parallel \vec{e}_{1(2)}$ near tip

Kinking - the classics



- crack tip singularity:

$$u = \sqrt{r} \sum_{i=1,2} \{K_i(t, l{+}l', \theta)\varphi_i\} + \hat{u}$$

with $\hat{u}$ smoother; φ_i universal fcts.

$=: u_{00}$(defined on all of $\mathbb{R}^2$) $+ \hat{u}$

- $K_{1(2)} = 0$ if $\sigma \vec{e}_2 \parallel \vec{e}_{1(2)}$ near tip

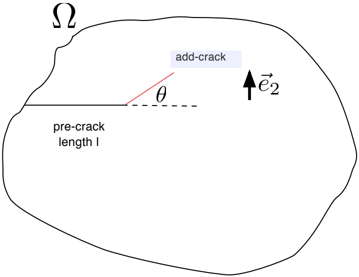
- shared by all:

$G(t, l) = k$ at time t when crack kinks $\approx$ energy conservation

- problem: what determines θ?
- 2 schools:

θ maximizes $G(t, l, \theta)$ *vs.* $0 = K_2^*(t, l, \theta) := \lim_{l' \searrow 0} K_2(t, l{+}l', \theta)$.

Kinking - the classics



- crack tip singularity:

$$u = \sqrt{r} \sum_{i=1,2} \{K_i(t, l{+}l', \theta)\varphi_i\} + \hat{u}$$

with $\hat{u}$ smoother; φ_i universal fcts.

$=: u_{00}$(defined on all of $\mathbb{R}^2$) $+ \hat{u}$

- $K_{1(2)} = 0$ if $\sigma \vec{e}_2 \parallel \vec{e}_{1(2)}$ near tip

- shared by all:
$G(t, l) = k$ at time t when crack kinks $\approx$ energy conservation
- problem: what determines θ?
- 2 schools:
θ maximizes $G(t, l, \theta)$ *vs.* $0 = K_2^*(t, l, \theta) := \lim_{l' \searrow 0} K_2(t, l{+}l', \theta)$.

$$\Uparrow$$

- Amestoy-Leblond 92: criteria do not coincide!

Revisiting energy release rates Chambolle-F-Marigo

- framework:
  - pre-crack $\gamma_i \approx$ straight near crack tip;
  - connected add-crack: $\Gamma_\varepsilon \xrightarrow{\text{Hausdorff}} \Gamma$;
  - boundary displacement u_0;
  - isotropic linear elasticity;
  - soln. to eqm. with γ_i: u_0

Revisiting energy release rates Chambolle-F-Marigo

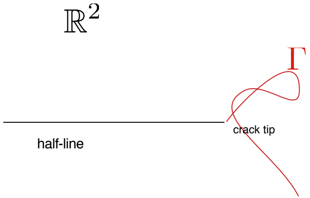
- framework:
  - pre-crack $\gamma_i \approx$ straight near crack tip;
  - connected add-crack: $\Gamma_\varepsilon \xrightarrow{\text{Hausdorff}} \Gamma$;
  - boundary displacement u_0;
  - isotropic linear elasticity;
  - soln. to eqm. with γ_i: u_0
- Blow up Thm: $1/\varepsilon\{\int_{\Omega\setminus(\gamma_i\cup\varepsilon\Gamma_\varepsilon)}\mathcal{C}e(u^{\varepsilon\Gamma_\varepsilon})\cdot e(u^{\varepsilon\Gamma_\varepsilon})dx - \int_{\Omega\setminus\gamma_i}\mathcal{C}e(u_0)\cdot e(u_0)dx\} \equiv$ energy release slope associated with add-crack $\varepsilon\Gamma_\varepsilon$

Revisiting energy release rates Chambolle-F-Marigo

- framework:
  - pre-crack $\gamma_i \approx$ straight near crack tip;
  - connected add-crack: $\Gamma_\varepsilon \xrightarrow{\text{Hausdorff}} \Gamma$;
  - boundary displacement u_0;
  - isotropic linear elasticity;
  - soln. to eqm. with γ_i: u_0
- Blow up Thm: $\lim_\varepsilon 1/\varepsilon\{\int_{\Omega\setminus(\gamma_i\cup\varepsilon\Gamma_\varepsilon)} \mathcal{C}e(u^{\varepsilon\Gamma_\varepsilon})\cdot e(u^{\varepsilon\Gamma_\varepsilon})dx - \int_{\Omega\setminus\gamma_i} \mathcal{C}e(u_0)\cdot e(u_0)dx\} = \mathcal{F}^\Gamma :=$
elast. energy release due to add-crack Γ starting from tip of straight half-line in dir. of pre-crack in $\mathbb{R}^2$

Revisiting energy release rates Chambolle-F-Marigo

- framework:
  - pre-crack $\gamma_i \approx$ straight near crack tip;
  - connected add-crack: $\Gamma_\varepsilon \xrightarrow{\text{Hausdorff}} \Gamma$;
  - boundary displacement u_0;
  - isotropic linear elasticity;
  - soln. to eqm. with γ_i: u_0
- Blow up Thm: $\lim_\varepsilon 1/\varepsilon\{\int_{\Omega\setminus(\gamma_i\cup\varepsilon\Gamma_\varepsilon)} \mathcal{C}e(u^{\varepsilon\Gamma_\varepsilon}) \cdot e(u^{\varepsilon\Gamma_\varepsilon})dx - \int_{\Omega\setminus\gamma_i} \mathcal{C}e(u_0)\cdot e(u_0)dx\} = \mathcal{F}^\Gamma :=$



Revisiting energy release rates Chambolle-F-Marigo

- framework:
  - pre-crack $\gamma_i \approx$ straight near crack tip;
  - connected add-crack: $\Gamma_\varepsilon \xrightarrow{\text{Hausdorff}} \Gamma$;
  - boundary displacement u_0;
  - isotropic linear elasticity;
  - soln. to eqm. with γ_i: u_0
- Blow up Thm: $\lim_\varepsilon 1/\varepsilon\{\int_{\Omega\setminus(\gamma_i\cup\varepsilon\Gamma_\varepsilon)} \mathcal{C}e(u^{\varepsilon\Gamma_\varepsilon}) \cdot e(u^{\varepsilon\Gamma_\varepsilon})dx - \int_{\Omega\setminus\gamma_i} \mathcal{C}e(u_0)\cdot e(u_0)dx\} = \mathcal{F}^\Gamma :=$

elast. energy release due to add-crack Γ starting from tip of straight half-line in dir. of pre-crack in $\mathbb{R}^2$

$$:= \min\{1/2\int_{\mathbb{R}^2} \mathcal{C}e(w)\cdot e(w)dx + \int_{B(0,r)} \mathcal{C}e(u_{00})\cdot e(w)dx - \int_{\partial B(0,r)} \mathcal{C}e(u_{00})\cdot(w\otimes\nu)d\mathcal{H}_1 : w \in H^1_{loc}(\mathbb{R}^2\setminus(\mathbb{R}^-\vec{e}_1\cup\Gamma)\}\square$$

↑ avoids dealing with infinite energies

Revisiting energy release rates **Chambolle-F-Marigo**

- framework:
  - pre-crack $\gamma_i \approx$ straight near crack tip;
  - connected add-crack: $\Gamma_\varepsilon \xrightarrow{\text{Hausdorff}} \Gamma$;
  - boundary displacement u_0;
  - isotropic linear elasticity;
  - soln. to eqm. with γ_i: u_0
- Blow up Thm: $\lim_\varepsilon 1/\varepsilon\{\int_{\Omega\setminus(\gamma_i\cup\varepsilon\Gamma_\varepsilon)} \mathcal{C}e(u^{\varepsilon\Gamma_\varepsilon}) \cdot e(u^{\varepsilon\Gamma_\varepsilon})dx - \int_{\Omega\setminus\gamma_i} \mathcal{C}e(u_0) \cdot e(u_0)dx\} = \mathcal{F}^\Gamma :=$

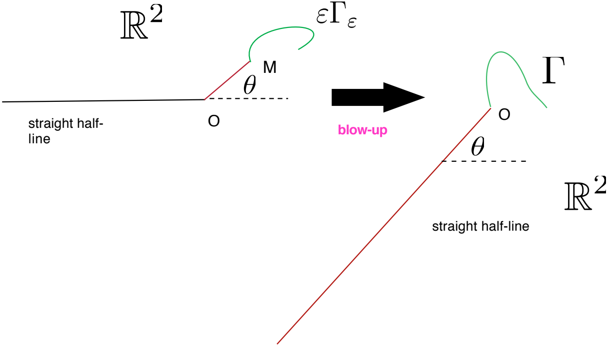
elast. energy release due to add-crack Γ starting from tip of straight half-line in dir. of pre-crack in $\mathbb{R}^2$

$$:= \min\{1/2 \int_{\mathbb{R}^2} \mathcal{C}e(w)\cdot e(w)dx + \int_{B(0,r)} \mathcal{C}e(u_{00})\cdot e(w)dx - \int_{\partial B(0,r)} \mathcal{C}e(u_{00})\cdot(w\otimes\nu)d\mathcal{H}_1 : w \in H^1_{loc}(\mathbb{R}^2\setminus(\mathbb{R}^-\vec{e}_1\cup\Gamma)\}\square$$

- Rk.: $\Gamma_\varepsilon = \Gamma(\varepsilon)/\varepsilon$ with $\Gamma(\varepsilon) \nearrow$ with ε, $\mathcal{H}_1(\Gamma(\varepsilon)) = \varepsilon$ and $\Gamma(\varepsilon)$ has density $1/2$ at 0, then $\Gamma_\varepsilon \xrightarrow{\text{Hausdorff}}$ unit length line-segment.

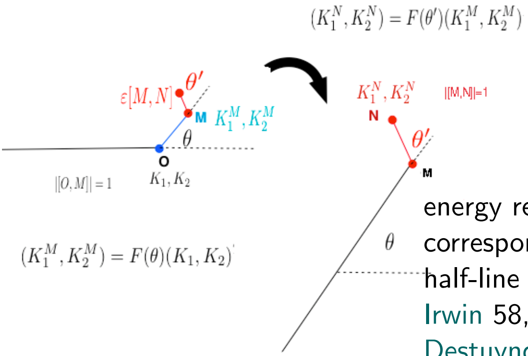
Revisiting energy release rates Chambolle-F-Marigo

- framework:
  - pre-crack $\gamma_i \approx$ straight near crack tip;
  - connected add-crack: $\Gamma_\varepsilon \xrightarrow{\text{Hausdorff}} \Gamma$;
  - boundary displacement u_0;
  - isotropic linear elasticity;
  - soln. to eqm. with γ_i: u_0
- Blow up Thm on $\mathbb{R}^2$:



Revisiting energy release rates II

SIF



energy release corresponding to half-line $+ [O, M]$, resp. $[O, N]$ Irwin 58, Destuynder-Djaoua 81:

$$\mathcal{C}((K_1^M)^2 + (K_2^M)^2), \text{ resp. } \mathcal{C}((K_1^N)^2 + (K_2^N)^2)$$

$$\Downarrow$$

- **Thm**: If $K_2 \neq 0$, then $\min_{\Gamma;\mathcal{H}^1(\Gamma)=1} \mathcal{F}^\Gamma$ is not attained for Γ unit-length line segments $\Rightarrow$ maximal energy release$>$ energy release rate for add-cracks with density $1/2$ $\square$

Theorem proved iff $\theta' = 0$ is not a maximum of en. release among all segments $[O, N]$ originating from O, assuming that $[O, M]$ attains the max. energy release.

Revisiting energy release rates III

- $F(\zeta)$ analytic universal matrix: expansion determined for small ζ's in Amestoy-Leblond 92

$$\theta_{max} \neq 0 \text{ if } \quad F_{21}(\zeta)F'_{12}(\zeta) - F_{22}(\zeta)F'_{11}(\zeta) \neq 0, \forall \zeta$$

$$\Downarrow$$

Among small ζ's, result is true.

- Conjecture numerically satisfied for large angles.

Consequence of meta-stability + energy conservation

• Assumptions of "generalized" classical kinking: existence of smooth evolution:

- Γ has density $1/2$ at crack tip;
- $\Gamma(t) \subset \Gamma$;
- $\Gamma(0) = \emptyset$;
- $\Gamma(t) \nearrow$ strictly and continuously in length;

$\Rightarrow$ energy release rate at $0 = k$

Consequence of meta-stability + energy conservation

• Assumptions of "generalized" classical kinking: existence of smooth evolution:

- Γ has density $1/2$ at crack tip;
- $\Gamma(t) \subset \Gamma$;
- $\Gamma(0) = \emptyset$;
- $\Gamma(t) \nearrow$ strictly and continuously in length;

$\Rightarrow$ energy release rate at $0 = k$

• Assume meta-stability: at $t = 0, \gamma_i$ locally minimizes total energy/length $\Rightarrow$ maximal energy release slope $\leq k$

Consequence of meta-stability + energy conservation

- Assumptions of "generalized" classical kinking: existence of smooth evolution:
  - Γ has density $1/2$ at crack tip;
  - $\Gamma(t) \subset \Gamma$;
  - $\Gamma(0) = \emptyset$;
  - $\Gamma(t) \nearrow$ strictly and continuously in length;

$\Rightarrow$ energy release rate at $0 = k$

- Assume meta-stability: at $t = 0, \gamma_i$ locally minimizes total energy/length $\Rightarrow$ maximal energy release slope $\leq k$
- But $\overset{\text{Thm+ Blow up Thm}}{\Longrightarrow}$ energy release rate at 0 along path $\Gamma(t) < k$!

$$\Downarrow$$

- Cor: No time continuous kinking onto add-cracks of density $1/2$

$\square$

Consequence of meta-stability + energy conservation

- Assumptions of "generalized" classical kinking: existence of smooth evolution:
  - Γ has density $1/2$ at crack tip;
  - $\Gamma(t) \subset \Gamma$;
  - $\Gamma(0) = \emptyset$;
  - $\Gamma(t) \nearrow$ strictly and continuously in length;

$\Rightarrow$ energy release rate at $0 = k$

- Assume meta-stability: at $t = 0, \gamma_i$ locally minimizes total energy/length $\Rightarrow$ maximal energy release slope $\leq k$
- But $\overset{\text{Thm+ Blow up Thm}}{\Longrightarrow}$ energy release rate at 0 along path $\Gamma(t) < k$!

$$\Downarrow$$

- Cor: No time continuous kinking onto add-cracks of density $1/2$

☐

- Either jump, or fork like pattern, or lack of connectedness!